

(c)

$$\begin{aligned}
& \frac{m^2 - mn}{6m^2 + 11mn + 3n^2} \div \frac{m^2 - n^2}{2m^2 - mn - 6n^2} \\
&= \frac{4m^2 - 7mn - 2n^2}{3m^2 + 7mn + 2n^2} \\
&= \frac{m(m-n)}{(3m+n)(2m+3n)} \div \frac{(m-n)(m+n)}{(2m+3n)(m-2n)} \div \frac{(4m+n)(m-2n)}{(3m+n)(m+2n)} \\
&\quad \text{for } m \neq -\frac{n}{3}, -\frac{3n}{2}, \pm 2n \\
&= \frac{m(m-n)}{(3m+n)(2m+3n)} \times \frac{(2m+3n)(m-2n)}{(m-n)(m+n)} \times \frac{(3m+n)(m+2n)}{(4m+n)(m-2n)} \\
&\quad \text{for } m \neq -\frac{n}{4}, -\frac{n}{3}, -\frac{3n}{2}, \pm n, \pm 2n \\
&= \frac{m}{1} \times \frac{1}{(m+n)} \times \frac{(m+2n)}{(4m+n)} \\
&= \frac{m(m+2n)}{(m+n)(4m+n)} \quad \text{or} \quad \frac{m^2 + 2mn}{4m^2 + 5mn + n^2}, \\
&\quad m \neq -\frac{n}{4}, -\frac{n}{3}, -\frac{3n}{2}, \pm n, \pm 2n \text{ or} \\
&\quad m \neq -0.25n, -0.3n, -1.5n, \pm n, \pm 2n
\end{aligned}$$

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1. (a) $1 + i$

(b) $-1 - 2i$

(c) $\sqrt{3} + i$

(d) $-2 - \sqrt{2}i$

(e) $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$

(f) $x - yi$

2. (a)

$$\begin{aligned}
\frac{1}{1-i} \times \frac{1+i}{1+i} &= \frac{1+i}{1-i^2} \\
&= \frac{1+i}{1-(-1)} \\
&= \frac{1}{2} - \frac{1}{2}i
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{1}{-1+2i} \times \frac{-1-2i}{-1-2i} &= \frac{-1-2i}{1-4i^2} \\
&= \frac{-1-2i}{1-4(-1)} \\
&= -\frac{1}{5} - \frac{2}{5}i
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{1}{\sqrt{3}-i} \times \frac{\sqrt{3}+i}{\sqrt{3}+i} &= \frac{\sqrt{3}+i}{3-i^2} \\
&= \frac{\sqrt{3}+i}{3-(-1)} \\
&= \frac{\sqrt{3}}{4} + \frac{1}{4}i
\end{aligned}$$

(d)

$$\begin{aligned}
\frac{1}{-2+\sqrt{2}i} \times \frac{-2-\sqrt{2}i}{-2-\sqrt{2}i} &= \frac{-2-\sqrt{2}i}{4-2i^2} \\
&= \frac{-2-\sqrt{2}i}{4-2(-1)} \\
&= -\frac{2}{6} - \frac{\sqrt{2}}{6}i \\
&= -\frac{1}{3} - \frac{\sqrt{2}}{6}i
\end{aligned}$$

(e)

$$\begin{aligned}
\frac{1}{-\frac{1}{2}-\frac{\sqrt{3}}{2}i} \times \frac{-\frac{1}{2}+\frac{\sqrt{3}}{2}i}{-\frac{1}{2}+\frac{\sqrt{3}}{2}i} &= \frac{-\frac{1}{2}+\frac{\sqrt{3}}{2}i}{\frac{1}{4}-\frac{3}{4}i^2} \\
&= \frac{-\frac{1}{2}+\frac{\sqrt{3}}{2}i}{\frac{1}{4}-\frac{3}{4}(-1)} \\
&= -\frac{1}{2} + \frac{\sqrt{3}}{2}i
\end{aligned}$$

(f)

$$\begin{aligned}
\frac{1}{x+yi} \times \frac{x-yi}{x-yi} &= \frac{x-yi}{x^2-y^2i^2} \\
&= \frac{x-yi}{x^2-y^2(-1)} \\
&= \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i
\end{aligned}$$

x, y not both zero.

3. (a)

$$\begin{aligned}
\frac{30}{3+i} \times \frac{3-i}{3-i} &= \frac{90-30i}{9-i^2} \\
&= \frac{90-30i}{9-(-1)} \\
&= 9-3i
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{2}{1+i} \times \frac{1-i}{1-i} &= \frac{2-2i}{1-i^2} \\
&= \frac{2-2i}{1-(-1)} \\
&= 1+i
\end{aligned}$$

(c)

$$\begin{aligned}\frac{4i}{1-i} \times \frac{1+i}{1+i} &= \frac{4i+4i^2}{1-i^2} \\ &= \frac{4i+4(-1)}{1-(-1)} \\ &= 2i-2 \\ &= -2+2i\end{aligned}$$

(d)

$$\begin{aligned}\frac{10i}{1-2i} \times \frac{1+2i}{1+2i} &= \frac{10i+20i^2}{1-4i^2} \\ &= \frac{10i+20(-1)}{1-4(-1)} \\ &= 2i-4 \\ &= -4+2i\end{aligned}$$

(e)

$$\begin{aligned}\frac{2+3i}{1-i} \times \frac{1+i}{1+i} &= \frac{2+5i+3i^2}{1-i^2} \\ &= \frac{2+5i+3(-1)}{1-(-1)} \\ &= \frac{-1+5i}{2} \\ &= -\frac{1}{2} + \frac{5}{2}i \text{ or } -0.5 + 2.5i\end{aligned}$$

(f)

$$\begin{aligned}\frac{3-i}{1+2i} \times \frac{1-2i}{1-2i} &= \frac{3-7i+2i^2}{1-4i^2} \\ &= \frac{3-7i+2(-1)}{1-4(-1)} \\ &= \frac{1-7i}{5} \\ &= \frac{1}{5} - \frac{7}{5}i \text{ or } 0.2 - 1.4i\end{aligned}$$

(g)

$$\begin{aligned}\frac{4i-3}{3+2i} \times \frac{3-2i}{3-2i} &= \frac{18i-8i^2-9}{9-4i^2} \\ &= \frac{18i-8(-1)-9}{9-4(-1)} \\ &= \frac{-1+18i}{13} \\ &= -\frac{1}{13} + \frac{18}{13}i\end{aligned}$$

(h)

$$\begin{aligned}\frac{4+i}{i} \times \frac{-i}{-i} &= \frac{-4i-i^2}{-i^2} \\ &= \frac{-4i-(-1)}{-(-1)} \\ &= 1-4i\end{aligned}$$

(i)

$$\begin{aligned}\frac{-2+5i}{-i} \times \frac{i}{i} &= \frac{-2i+5i^2}{-i^2} \\ &= \frac{-2i+5(-1)}{-(-1)} \\ &= -5-2i\end{aligned}$$

(j)

$$\begin{aligned}\frac{x+yi}{x-yi} \times \frac{x+yi}{x+yi} &= \frac{x^2+2xyi+y^2i^2}{x^2-y^2i^2}, x, y \text{ not both zero} \\ &= \frac{x^2+2xyi+y^2(-1)}{x^2-y^2(-1)} \\ &= \frac{x^2-y^2}{x^2+y^2} + \frac{2xy}{x^2+y^2}i\end{aligned}$$

4.

$$z_1 = \sqrt{3} + i, z_2 = \sqrt{3} - i$$

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{\sqrt{3}+i}{\sqrt{3}-i} \times \frac{\sqrt{3}+i}{\sqrt{3}+i} \\ &= \frac{3+2\sqrt{3}i+i^2}{3-i^2} \\ &= \frac{3+2\sqrt{3}i+(-1)}{3-(-1)} \\ &= \frac{2+2\sqrt{3}i}{4} \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}i \text{ or } 0.5 + 0.87i\end{aligned}$$

5.

$$z = 3-i, w = 1+2i$$

$$\begin{aligned}\frac{z}{2w} &= \frac{3-i}{2(1+2i)} \times \frac{1-2i}{1-2i} \\ &= \frac{3-7i+2i^2}{2(1-4i^2)} \\ &= \frac{3-7i+2(-1)}{2(1-4(-1))} \\ &= \frac{1-7i}{10} \\ &= \frac{1}{10} - \frac{7}{10}i \text{ or } 0.1 - 0.7i\end{aligned}$$

6.

$$\begin{aligned}
 z_1 &= 1 - i, \quad z_2 = 7 + i \\
 \frac{z_1 - z_2}{z_1 z_2} &= \frac{1 - i - (7 + i)}{(1 - i)(7 + i)} \\
 &= \frac{1 - i - 7 - i}{7 - 6i - i^2} \\
 &= \frac{-6 - 2i}{7 - 6i - (-1)} \\
 &= \frac{-6 - 2i}{8 - 6i} \\
 &= \frac{-2(3 + i)}{2(4 - 3i)} \times \frac{4 + 3i}{4 + 3i} \\
 &= \frac{-(12 + 13i + 3i^2)}{16 - 9i^2} \\
 &= \frac{-(12 + 13i + 3(-1))}{16 - 9(-1)} \\
 &= \frac{-9 - 13i}{25} \\
 &= -\frac{9}{25} - \frac{13}{25}i
 \end{aligned}$$

7.

$$\begin{aligned}
 \frac{1 - i}{(3 - i)^2} &= \frac{1 - i}{9 - 6i + i^2} \\
 &= \frac{1 - i}{9 - 6i + (-1)} \\
 &= \frac{1 - i}{8 - 6i} \\
 &= \frac{1 - i}{2(4 - 3i)} \times \frac{4 + 3i}{4 + 3i} \\
 &= \frac{4 - i - 3i^2}{2(16 - 9i^2)} \\
 &= \frac{4 - i - 3(-1)}{2(16 - 9(-1))} \\
 &= \frac{7 - i}{2(25)} \\
 &= \frac{7}{50} - \frac{1}{50}i
 \end{aligned}$$

8. The method of dividing real numbers can be compared to the method of dividing complex numbers when we think of division of real numbers as multiplication by the reciprocal. We simplify a quotient of complex numbers by making the denominator real. This is achieved by multiplying the quotient by an expression equivalent to 1 in which the numerator and denominator are both the complex conjugate of the denominator.

Division by a real number can be thought of in a similar way, if it is achieved by multiplying the quotient of real numbers by an expression, equivalent to 1, in which the numerator and denominator are both the reciprocal of the denominator. In this way, the denominator becomes 1.

Examples:

Division of complex numbers:

$$\begin{aligned}
 \frac{2 + i}{4 - i} &\times \frac{4 + i}{4 + i} \quad (\text{complex conjugate of denominator}) \\
 &= \frac{8 + 6i + i^2}{16 - i^2} \\
 &= \frac{7 + 6i}{17} \quad (\text{denominator is a real number})
 \end{aligned}$$

Division of real numbers:

$$\begin{aligned}
 \frac{3}{5} &\times \frac{1}{\frac{1}{5}} \quad (\text{reciprocal of denominator}) \\
 &= \frac{3}{1} \quad (\text{denominator is the number 1})
 \end{aligned}$$

9. It is important to remember that i is not a variable, and so when faced with a quotient such as $\frac{2 + i}{4 - i}$ there is no possibility that $4 - i$ is zero, since i always represents the same number, and is not equal to 4. However, if dealing with an expression such as $\frac{a + ib}{c + id}$, the expression would not be valid if both c and d were zero. Therefore, the restriction that c and d are not both zero must be placed. It is possible for either c or d to be zero, but not both at the same time.

10. (a) If $z = 3 + 4i$,

$$\begin{aligned}
 z + \frac{25}{z} &= 3 + 4i + \frac{25}{3 + 4i} \times \frac{3 - 4i}{3 - 4i} \\
 &= 3 + 4i + \frac{25(3 - 4i)}{9 - 16i^2} \\
 &= 3 + 4i + \frac{25(3 - 4i)}{25} \\
 &= 3 + 4i + 3 - 4i \\
 &= 6 + 0i
 \end{aligned}$$

(b) If $z = a + bi$,

$$\begin{aligned}
 z + \frac{1}{z} &= a + bi + \frac{1}{a + bi} \times \frac{a - bi}{a - bi}, \quad a, b \text{ not both zero} \\
 &= a + bi + \frac{a - bi}{a^2 - b^2 i^2} \\
 &= a + bi + \frac{a - bi}{a^2 + b^2} \\
 &= a + bi + \frac{a}{a^2 + b^2} - \frac{bi}{a^2 + b^2} \\
 &= a + \frac{a}{a^2 + b^2} + bi - \frac{bi}{a^2 + b^2}
 \end{aligned}$$

Real part is $a + \frac{a}{a^2 + b^2}$, imaginary part is $\left(b - \frac{b}{a^2 + b^2}\right)i$.

11. (a)

$$(5-2i) = z(1-i)$$

$$\frac{5-2i}{1-i} = z$$

$$\frac{5-2i}{1-i} \times \frac{1+i}{1+i} = z$$

$$\frac{5+3i-2i^2}{1-i^2} = z$$

$$\frac{5+3i-2(-1)}{1-(-1)} = z$$

$$\frac{7+3i}{2} = z$$

$$z = \frac{7}{2} + \frac{3}{2}i$$

$$= 3.5 + 1.5i$$

(b)

$$(3+2i) + z = (1-i)z$$

$$3+2i = z - iz - z$$

$$3+2i = -iz$$

$$\frac{3+2i}{-i} = z$$

$$\frac{3+2i}{-i} \times \frac{i}{i} = z$$

$$\frac{3i+2i^2}{-i^2} = z$$

$$3i-2 = z$$

$$z = -2 + 3i$$

(c)

$$4z - z(3+i) = -1+3i$$

$$4z - 3z - iz = -1+3i$$

$$z - iz = -1+3i$$

$$z(1-i) = -1+3i$$

$$z = \frac{-1+3i}{1-i} \times \frac{1+i}{1+i}$$

$$= \frac{-1+2i+3i^2}{1-i^2}$$

$$= \frac{-4+2i}{2}$$

$$= -2+i$$

(d)

$$z(5-3i)^2 = 4+3i$$

$$z(25-30i+9i^2) = 4+3i$$

$$z(25-30i-9) = 4+3i$$

$$z(16-30i) = 4+3i$$

$$z = \frac{4+3i}{16-30i} \times \frac{16+30i}{16+30i}$$

$$= \frac{(4+3i)(16+30i)}{256-900i^2}$$

$$= \frac{64+120i+48i+90i^2}{256+900}$$

$$= \frac{64+168i-90}{1156}$$

$$= \frac{-26+168i}{1156}$$

$$= \frac{-13}{578} + \frac{42}{289}i$$

12. (a)

$$\frac{2}{3+i} + \frac{3}{2+i} = \frac{2}{3+i} \times \frac{3-i}{3-i} + \frac{3}{2+i} \times \frac{2-i}{2-i}$$

$$= \frac{6-2i}{9-i^2} + \frac{6-3i}{4-i^2}$$

$$= \frac{6-2i}{9-(-1)} + \frac{6-3i}{4-(-1)}$$

$$= \frac{6-2i}{10} + \frac{6-3i}{5}$$

$$= \frac{3}{5} - \frac{1}{5}i + \frac{6}{5} - \frac{3}{5}i$$

$$= \frac{9}{5} - \frac{4}{5}i \text{ or } 1.8 - 0.8i$$

(b)

$$\frac{1}{a+bi} - \frac{1}{a-bi}$$

$$= \frac{1}{a+bi} \times \frac{a-bi}{a-bi} - \frac{1}{a-bi} \times \frac{a+bi}{a+bi}$$

$$= \frac{a-bi}{a^2-b^2i^2} - \frac{a+bi}{a^2-b^2i^2}$$

$$= \frac{a-bi}{a^2+b^2} - \frac{a+bi}{a^2+b^2}$$

$$= \frac{a}{a^2+b^2} - \frac{bi}{a^2+b^2} - \frac{a}{a^2+b^2} - \frac{bi}{a^2+b^2}$$

$$= 0 - \frac{2bi}{a^2+b^2}$$

$$= 0 - \frac{2b}{a^2+b^2}i$$

$$13. \quad z_1 = \frac{a}{1+i}, \quad z_2 = \frac{b}{1+2i}$$

$$z_1 + z_2 = 1$$

$$\frac{a}{1+i} + \frac{b}{1+2i} = 1$$

$$\frac{a}{1+i} \times \frac{1-i}{1-i} + \frac{b}{1+2i} \times \frac{1-2i}{1-2i} = 1$$

$$\frac{a(1-i)}{1-i^2} + \frac{b(1-2i)}{1-4i^2} = 1$$

$$\frac{a-ai}{2} + \frac{b-2bi}{5} = 1$$

$$\frac{a}{2} - \frac{a}{2}i + \frac{b}{5} - \frac{2}{5}bi = 1$$

$$\frac{a}{2} + \frac{b}{5} - \frac{a}{2}i - \frac{2}{5}bi = 1$$

Multiply by 10.

$$10\left(\frac{a}{2}\right) + 10\left(\frac{b}{5}\right) - 10\left(\frac{a}{2}i\right) - 10\left(\frac{2b}{5}i\right) = 10(1)$$

$$5a + 2b - 5ai - 4bi = 10$$

$$5a + 2b - (5a + 4b)i = 10 + 0i$$

Equate the real parts on each side of the equation.

$$5a + 2b = 10 \text{ -----①}$$

Equate the imaginary parts.

$$-5a - 4b = 0 \text{ -----②}$$

Add (1) + (2):

$$0a - 2b = 10$$

$$b = -5$$

Substitute $b = -5$ into ①:

$$5a + 2(-5) = 10$$

$$5a = 20$$

$$a = 4$$

14.

$$\frac{z}{z+2} = 2-i$$

$$z = (2-i)(z+2)$$

$$z = 2z + 4 - iz - 2i$$

$$z - 2z + iz = 4 - 2i$$

$$-z + iz = 4 - 2i$$

$$z(-1+i) = 4 - 2i$$

$$z = \frac{4-2i}{-1+i} \times \frac{-1-i}{-1-i}$$

$$= \frac{-4-2i+2i^2}{-1-i^2}$$

$$= \frac{-4-2i+2(-1)}{1-(-1)}$$

$$= \frac{-6-2i}{2}$$

$$= -3-i$$

15. $z\bar{z}$ is always a real number. Therefore, if $\frac{w}{z}$ is a quotient of complex numbers w and z , $\frac{w\bar{z}}{z\bar{z}}$ will have a real number in

the denominator and will be in the form $a + bi$. For example, the quotient $\frac{3-i}{2+i}$ is the same as

$$\frac{3-i}{2+i} \times \frac{2-i}{2-i} = \frac{6-5i+i^2}{4-i^2}$$

$$= \frac{6-5i+(-1)}{4-(-1)}$$

$$= \frac{5-5i}{5} \quad (\text{the denominator is now a real number})$$

$$= 1-i$$

16.

$$z + (1+i)w = 1 - 2i \text{ -----①}$$

$$(1+2i)z - 2iw = 9 + 6i \text{ -----②}$$

Multiply equation ① by $(1+2i)$:

$$(1+2i)z + (1+2i)(1+i)w = (1+2i)(1-2i)$$

$$(1+2i)z + (1+3i+2i^2)w = 1-4i^2$$

$$(1+2i)z + (-1+3i)w = 5 \text{ -----③}$$

$$(1+2i)z - 2iw = 9 + 6i \text{ -----②}$$

Subtract ③ - ②:

$$0z + (-1+3i+2i)w = -4-6i$$

$$(-1+5i)w = -4-6i$$

$$w = \frac{-4-6i}{-1+5i} \times \frac{-1-5i}{-1-5i}$$

$$= \frac{4+26i+30i^2}{1-25i^2}$$

$$= \frac{-26+26i}{26}$$

$$w = -1+i$$

Substitute $w = -1+i$ into equation ①:

$$z + (1+i)(-1+i) = 1-2i$$

$$z + (-1+i^2) = 1-2i$$

$$z - 2 = 1-2i$$

$$z = 1-2i+2$$

$$z = 3-2i$$

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1. (a) $\frac{3}{x}, \frac{5}{y}$;

LCD is xy .

(b) $\frac{-2}{ab}, \frac{1}{b}$;

LCD is ab .

(c) $\frac{4}{x-1}, \frac{2}{x}$;

LCD is $x(x-1)$.

(d) $\frac{b}{b+2}, \frac{3b}{b+1}$;

LCD is $(b+1)(b+2)$.

(e) $\frac{5}{x^2}, \frac{7}{xy}$;

LCD is x^2y .

(f) $\frac{m-1}{2m+3}, \frac{m}{m-3}$;

LCD is $(m-3)(2m+3)$.

(g) $\frac{11x}{yz^2}, \frac{8x}{y^2z}$;

LCD is y^2z^2 .

(h) $\frac{3s}{(s-1)(s+1)}, \frac{s+2}{s-1}$;

LCD is $(s-1)(s+1)$.