

24. $300 - 299 + 298 - 297 + 296 - \dots + 100 - 99$ can be rewritten as:
 $(300 + 298 + 296 + \dots + 100) - (299 + 297 + 295 + \dots + 99)$.
 We find the number of terms in either series as they both have the same number of terms. We solve for n with $a = 300$,
 $d = -2$, and $t_n = 100$.
 $t_n = a + (n - 1)d$
 $100 = 300 + (n - 1)(-2)$
 $100 = 300 - 2n + 2$
 $100 - 302 = -2n$
 $-202 = -2n$
 $n = 101$
 Since every term in the first series is one greater than each term in the second series, the total sum is $101 \times 1 = 101$.

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1. (a) i. geometric, **ii.** $r = \frac{1}{2}$

(b) i. geometric, **ii.** $r = \frac{4}{5}$

(c) i. neither

(d) i. geometric, **ii.** $r = -\frac{3}{2}$

(e) i. geometric, **ii.** $r = -\frac{1}{2}$

(f) i. neither

2. $t_n = ar^{n-1}$, $S_n = \frac{a(r^n - 1)}{r - 1}$

(a) i. $t_n = 2(3)^{n-1}$

ii.

$$S_n = \frac{2(3^n - 1)}{3 - 1}$$

$$= 2(3^n - 1)$$

iii. $S_8 = 2(3^8 - 1) = 2(6560) = 13\,120$

(b) i. $t_n = 1\left(-\frac{2}{3}\right)^{n-1} = \left(-\frac{2}{3}\right)^{n-1}$

ii.

$$S_n = \frac{1\left[\left(-\frac{2}{3}\right)^n - 1\right]}{-\frac{2}{3} - 1}$$

$$= \frac{\left(-\frac{2}{3}\right)^n - 1}{-\frac{5}{3}}$$

$$= -\frac{3}{5}\left[\left(-\frac{2}{3}\right)^n - 1\right]$$

iii.

$$S_8 = -\frac{3}{5}\left[\left(-\frac{2}{3}\right)^8 - 1\right]$$

$$= -\frac{3}{5}\left(\frac{256}{6561} - \frac{6561}{6561}\right)$$

$$= -\frac{3}{5}\left(-\frac{6305}{6561}\right)$$

$$= \frac{18\,915}{32\,805}$$

$$= \frac{1261}{2187}$$

$$\doteq 0.58$$

(c) i. $t_n = 6(-2)^{n-1}$

ii.

$$S_n = \frac{6[(-2)^n - 1]}{-2 - 1}$$

$$= -2[(-2)^n - 1]$$

iii. $S_8 = -2[(-2)^8 - 1] = -2(256 - 1) = -510$

(d) i. $t_n = 81\left(\frac{1}{3}\right)^{n-1}$

ii.

$$S_n = \frac{81\left[\left(\frac{1}{3}\right)^n - 1\right]}{\frac{1}{3} - 1}$$

$$= 81\left[\left(\frac{1}{3}\right)^n - 1\right] \times \left(-\frac{3}{2}\right)$$

$$= -\frac{243}{2}\left[\left(\frac{1}{3}\right)^n - 1\right]$$

iii.

$$S_8 = -\frac{243}{2}\left[\left(\frac{1}{3}\right)^8 - 1\right]$$

$$= -\frac{243}{2}\left(\frac{1}{6561} - \frac{6561}{6561}\right)$$

$$= -\frac{243}{2}\left(-\frac{6560}{6561}\right)$$

$$= \frac{3280}{27}$$

$$= 121.48$$

(e) i. $t_n = 0.4\left(\frac{1}{10}\right)^{n-1}$

ii.

$$S_n = \frac{0.4 \left[\left(\frac{1}{10} \right)^n - 1 \right]}{\frac{1}{10} - 1}$$

$$= \frac{\frac{4}{10} \left[\left(\frac{1}{10} \right)^n - 1 \right]}{-\frac{9}{10}}$$

$$= -\frac{4}{9} \left[\left(\frac{1}{10} \right)^n - 1 \right]$$

iii.

$$S_8 = -\frac{4}{9} \left[\left(\frac{1}{10} \right)^8 - 1 \right]$$

$$= -\frac{4}{9} \left(\frac{-99\,999\,999}{100\,000\,000} \right)$$

$$= -\frac{44\,444\,444}{100\,000\,000}$$

$$= 0.44$$

(f) i. $t_n = 8(-1)^{n-1}$

ii. $S_n = \frac{8[(-1)^n - 1]}{-1 - 1} = -4[(-1)^n - 1]$

iii. $S_8 = -4[(-1)^8 - 1] = -4(1 - 1) = 0$

3. $S_n = \frac{a(r^n - 1)}{r - 1}$

(a)

$$S_7 = \frac{5[(-2)^7 - 1]}{7 - 1}$$

$$= \frac{5(-129)}{6}$$

$$= -107.5$$

(b)

$$S_{10} = \frac{54 \left[\left(\frac{1}{3} \right)^{10} - 1 \right]}{\frac{1}{3} - 1}$$

$$= \frac{54 \left(\frac{-59\,048}{59\,049} \right)}{-\frac{2}{3}}$$

$$= -81 \left(\frac{-59\,048}{59\,049} \right)$$

$$= 81.00$$

(c)

$$S_8 = \frac{-300[(-15)^8 - 1]}{-15 - 1}$$

$$= \frac{-300(2\,562\,890\,624)}{-16}$$

$$= 48\,054\,199\,200 \text{ or } 4.805 \times 10^{10}$$

(d)

$$S_9 = \frac{2[(\sqrt{2})^9 - 1]}{\sqrt{2} - 1}$$

$$= \frac{2(21.627\,417)}{0.414\,213\,562\,4}$$

$$= 104.43$$

(e)

$$S_n = \frac{1(x^n - 1)}{x - 1}$$

$$= \frac{x^n - 1}{x - 1}$$

(f) $S_n = \frac{5w[(2w)^n - 1]}{2w - 1}$

4. (a) $t_n = ar^{n-1}$
 $t_8 = 6(-3)^7 = -13\,122$

(b)

$$S_8 = \frac{6[(-3)^8 - 1]}{-3 - 1}$$

$$= \frac{6(6560)}{-4}$$

$$= -9840$$

5. (a) $a = 7, r = 2, t_n = 3584$

$$3584 = 7(2)^{n-1}$$

$$512 = (2)^{n-1}$$

$$2^9 = 2^{n-1}$$

$$9 = n - 1$$

$$n = 10$$

$$S_{10} = \frac{7(2^{10} - 1)}{2 - 1}$$

$$= 7(1023)$$

$$= 7161$$

(b) $a = -3, r = 2, t_n = -768$

$$-768 = -3(2)^{n-1}$$

$$256 = 2^{n-1}$$

$$2^8 = 2^{n-1}$$

$$8 = n - 1$$

$$n = 9$$

$$S_9 = \frac{-3(2^9 - 1)}{2 - 1}$$

$$= -3(511)$$

$$= -1533$$

$$\begin{aligned}
 \text{(c)} \quad a &= 1, r = \frac{5}{2}, t_n = \frac{15\,625}{64} \\
 \frac{15\,625}{64} &= 1\left(\frac{5}{2}\right)^{n-1} \\
 \left(\frac{5}{2}\right)^6 &= \left(\frac{5}{2}\right)^{n-1} \\
 6 &= n-1 \\
 n &= 7 \\
 S_7 &= \frac{1\left[\left(\frac{5}{2}\right)^7 - 1\right]}{\frac{5}{2} - 1} \\
 &= \frac{78\,125 - 128}{\frac{3}{2}} \\
 &= \frac{77\,997}{\frac{3}{2}} \times \frac{2}{3} \\
 &= \frac{25\,999}{64} \text{ or } \doteq 406.234
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad a &= 96\,000, r = -\frac{1}{2}, t_n = 375 \\
 375 &= 96\,000\left(-\frac{1}{2}\right)^{n-1} \\
 \frac{375}{96\,000} &= \left(-\frac{1}{2}\right)^{n-1} \\
 \frac{1}{256} &= \left(-\frac{1}{2}\right)^{n-1} \\
 \left(-\frac{1}{2}\right)^8 &= \left(-\frac{1}{2}\right)^{n-1} \\
 8 &= n-1 \\
 n &= 9 \\
 S_9 &= \frac{96\,000\left[\left(-\frac{1}{2}\right)^9 - 1\right]}{-\frac{1}{2} - 1} \\
 &= \frac{96\,000\left(-\frac{1}{512} - \frac{512}{512}\right)}{-\frac{3}{2}} \\
 &= 96\,000\left(-\frac{2}{3}\right)\left(-\frac{513}{512}\right) \\
 &= -64\,000\left(-\frac{513}{512}\right) \\
 &= 64\,125
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad a &= 1000, r = 1.06, n = 13 \\
 S_{13} &= \frac{1000[(1.06)^{13} - 1]}{1.06 - 1} \\
 &= \frac{1000(1.132\,928\,26)}{0.06} \\
 &\doteq 18\,882.137\,67
 \end{aligned}$$

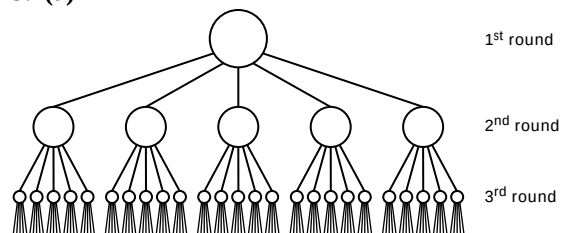
$$\begin{aligned}
 6. \quad t_5 &= 405 \text{ and } t_6 = 1215 \\
 r &= \frac{t_6}{t_5} = \frac{1215}{405} = 3 \\
 t_5 &= ar^4 \\
 405 &= a(3)^4 \\
 \frac{405}{81} &= a \\
 a &= 5 \\
 S_9 &= \frac{5(3^9 - 1)}{3 - 1} \\
 &= \frac{5(19\,682)}{2} \\
 &= 49\,205
 \end{aligned}$$

$$\begin{aligned}
 7. \quad a &= 3, r = 3, S_n = 9840 \\
 9840 &= \frac{3(3^n - 1)}{3 - 1} \\
 (9840)(2) &= 3(3^n - 1) \\
 \frac{19\,680}{3} &= (3^n - 1) \\
 6560 &= 3^n - 1 \\
 6561 &= 3^n \\
 3^8 &= 3^n \\
 n &= 8
 \end{aligned}$$

Therefore, it will take 8 rounds of phone calls.

8. Since his pay increases by 10% each month, his salary each month is 1.10 times greater than the previous month. His monthly salary can be represented by the geometric sequence $t_n = 1200(1.10)^{n-1}$ for the n th month. To calculate his total pay for the last 6 months of his first year, we can ① find the geometric sum from $n = 1$ to $n = 12$ and then subtract the geometric sum from $n = 1$ to $n = 6$ or ② find the geometric sum from $n = 7$ to $n = 12$.

9. (a)



(b) (i) For the first round of e-mails, 5 people will receive a copy of the joke.

(ii) For the second round of e-mails, 25 people will receive a copy of the joke.

(iii) For the third round of e-mails, 125 people will receive a copy of the joke.

(c) The total number of people that receive the joke after n mailings can be represented by a geometric series with $a = 5$ and $r = 5$.

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{5(5^n - 1)}{4}$$

$$\begin{aligned} \text{(d)} \quad S_8 &= \frac{5(5^8 - 1)}{4} \\ &= \frac{5(390\,624)}{4} \\ &= 488\,280 \end{aligned}$$

Therefore, 488 280 people would have received the e-mail after eight rounds. This is not likely to occur since not all of the receivers of the e-mail would forward it to 5 people. Also, it is likely that several people would receive the e-mail more than once since many receivers will forward it to some of their common friends or relatives.

10. Each second, the r/s is $\frac{2}{3}$ of the previous r/s . Therefore, the total turn the saw makes is represented by a geometric series with $r = \frac{2}{3}$ and $S_9 = 258$. Solve for a .

$$\begin{aligned} 258 &= \frac{a\left[\left(\frac{2}{3}\right)^9 - 1\right]}{\frac{2}{3} - 1} \\ 258 &= \frac{a\left(\frac{512}{19\,683} - \frac{19\,683}{19\,683}\right)}{-\frac{1}{3}} \\ 258\left(-\frac{3}{1}\right) &= a\left(-\frac{19\,171}{19\,683}\right) \\ a &= -774\left(-\frac{19\,683}{19\,171}\right) \\ a &= 794.67 \end{aligned}$$

Therefore, the speed of the saw was about 794.7 r/s when it was first shut off.

11. (a) $a = 19\,500$, $r = 1.02$

$$\begin{aligned} t_n &= ar^{n-1} \\ t_7 &= 19\,500(1.02)^6 = 21\,960.17 \end{aligned}$$

(b)

$$\begin{aligned} S_n &= \frac{a(r^n - 1)}{r - 1} \\ S_{10} &= \frac{19\,500(1.02^{10} - 1)}{1.02 - 1} \\ &= \frac{4270.391\,19}{0.02} \\ &= 213\,519.56 \end{aligned}$$

12. (a) Each subsequent year the company's profit is 1.15 times the previous year's profit. We find t_7 with $a = 245\,000$ and $r = 1.15$.

$$t_7 = 245\,000(1.15)^6 = 566\,699.8876$$

The company can expect a profit of \$566 699.89 or about \$566 700 in its seventh year.

(b) We find S_{10} with $a = 245\,000$ and $r = 1.15$.

$$\begin{aligned} S_{10} &= \frac{245\,000[(1.15)^{10} - 1]}{1.15 - 1} \\ &= \frac{245\,000(3.045\,557\,736)}{0.15} \\ &= 4\,974\,410.97 \end{aligned}$$

Therefore, the company can expect to make a total profit of \$4 974 410.97 over the first 10 years.

13. The downward distance is S_5 , the geometric series with $a = 15$ and $r = 0.7$.

$$\begin{aligned} S_n &= \frac{a(r^n - 1)}{r - 1} \\ S_5 &= \frac{15[(0.7)^5 - 1]}{0.7 - 1} \\ &= \frac{15(-0.831\,93)}{-0.3} \\ &\doteq 41.60 \end{aligned}$$

The total downward distance is 41.60 m and the upward distance is $(41.60 - 15)$ 26.6 m. The total distance travelled by the ball is 68.20 m $(41.60 + 26.60)$.

14. (a) We find t_5 of the geometric sequence having

$$a = 4500 \text{ and } r = 1.12.$$

$$t_5 = 4500(1.12)^4 = 7080.837\,12$$

They would need to raise \$7080.84 five years from now.

(b) Find S_7 of the geometric series having $a = 4500$ and $r = 1.12$.

$$\begin{aligned} S_7 &= \frac{4500[(1.12)^7 - 1]}{1.12 - 1} \\ &= \frac{4500(1.210\,681\,407)}{0.12} \\ &\doteq 45\,400.55 \end{aligned}$$

Therefore, they can expect to raise a total of \$45 400.55 after 7 years.

$$\begin{aligned}
15. rS_n &= a(r) + ar(r) + ar^2(r) + \dots + ar^{n-2}(r) + ar^{n-1}(r) \\
rS_n &= ar + ar^2 + ar^3 + \dots + ar^{n-1} + ar^n \\
S_n &= a + ar + ar^2 + \dots + ar^{n-2} + ar^{n-1} \\
rS_n - S_n &= ar^n - a \\
S_n(r-1) &= a(r^n - 1) \\
S_n &= \frac{a(r^n - 1)}{r - 1}
\end{aligned}$$

16. (a) The total prize money of \$4 000 000 is comprised of the sum of all the terms of the geometric sequence having $a = 15$ and $r = 3$. We must solve $S_n = 4\,000\,000$ for n .

$$\begin{aligned}
S_n &= \frac{a(r^n - 1)}{r - 1} \\
4\,000\,000 &= \frac{15(3^n - 1)}{3 - 1} \quad \text{Multiply both sides by 2.} \\
8\,000\,000 &= 15(3^n - 1) \quad \text{Divide both sides by 15.} \\
3^n - 1 &= 533\,333.33 \quad \text{Add 1 to each side.} \\
3^n &= 533\,334.33 \\
\ln(3^n) &= \ln(533\,334.33) \\
n \ln(3) &= \ln(533\,334.33) \\
n &= \frac{\ln(533\,334.33)}{\ln(3)} \\
n &= 12.003
\end{aligned}$$

(b) The total value of the prizes awarded is S_{12} .

$$\begin{aligned}
S_{12} &= \frac{15(3^{12} - 1)}{3 - 1} \\
&= \frac{15(531\,440)}{2} \\
&= 3\,985\,800
\end{aligned}$$

Therefore, (\$4 000 000 – \$3 985 800) or \$14 200 is left over after the 12 prizes have been awarded.

17. (a) Since the series has a common ratio of 4, we multiply each term in the series by 4.

$$\begin{aligned}
4S_7 &= 8 + 32 + 128 + 512 + 2048 + 8192 + 32\,768 \\
S_7 &= 2 + 8 + 32 + 128 + 512 + 2048 + 8192 \\
\hline
4S_7 - S_7 &= 32\,768 - 2 \\
3S_7 &= 32\,766 \\
S_7 &= \frac{32\,766}{3} \\
S_7 &= 10\,922
\end{aligned}$$

(b)

$$\begin{aligned}
S_n &= \frac{a(r^n - 1)}{r - 1} \\
S_7 &= \frac{2(4^7 - 1)}{4 - 1} \\
&= \frac{2(16\,383)}{3} \\
&= 10\,922
\end{aligned}$$

$$18. \text{ (a) } t_1 = 3, t_2 = 3r, t_3 = 3r^2$$

$$\text{ (b) } S_3 = t_1 + t_2 + t_3$$

$$21 = 3 + 3r + 3r^2$$

$$3r^2 + 3r - 18 = 0$$

$$3(r^2 + r - 6) = 0$$

$$3(r + 3)(r - 2) = 0$$

$$r = -3 \text{ and } r = 2$$

If $r = -3$, then $t_1 = 3$, $t_2 = -9$, $t_3 = 27$, and

$$S_3 = 3 - 9 + 27 = 21.$$

If $r = 2$, then $t_1 = 3$, $t_2 = 6$, $t_3 = 12$, and

$$S_3 = 3 + 6 + 12 = 21.$$

19. We find an expression for the geometric series S_n with $a = 2$ and $r = \frac{1}{2}$.

$$\begin{aligned}
S_n &= \frac{2\left[\left(\frac{1}{2}\right)^n - 1\right]}{\frac{1}{2} - 1} \\
&= \frac{2\left[\left(\frac{1}{2}\right)^n - 1\right]}{-\frac{1}{2}} \\
&= -4\left[\left(\frac{1}{2}\right)^n - 1\right]
\end{aligned}$$

As n gets larger and larger, $\left(\frac{1}{2}\right)^n$ gets closer and closer to

zero. When n is very very large we can replace $\left(\frac{1}{2}\right)^n$ by zero

or a very small positive number.

$$S_n = -4(0 - 1) = 4$$

But $\left(\frac{1}{2}\right)^n$ is never equal to zero. Therefore, $\left[\left(\frac{1}{2}\right)^n - 1\right]$ can

never be less than -1 . Therefore, $-4\left[\left(\frac{1}{2}\right)^n - 1\right]$ can never be equal to or greater than 4.

20. (a) $2 - 4 + 6 - 8 + 10 - \dots - 100$ can be rewritten as $(2 + 6 + 10 + \dots + 98) - (4 + 8 + 12 + \dots + 100)$.

Both arithmetic series have the same number of terms. We solve for n using $t_n = a + (n - 1)d$ with $a = 2$, $d = 4$, and $t_n = 98$.

$$98 = 2 + (n - 1)(4)$$

$$98 = 2 + 4n - 4$$

$$98 = 4n - 2$$

$$100 = 4n$$

$$n = 25$$

Each series has 25 terms. Since each term in the second series is 2 greater than each term in the first series, the sum of the differences of all the terms will be $(-25)(2) = -50$. Therefore the sum of the original series is -50 .

(b) The series is the sum of the numbers from 1 to 98 with the exclusion of the numbers 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48, 51, 54, 57, 60, 63, 66, 69, 72, 75, 78, 81, 84, 87, 90, 93, 96, and 99.

We find the arithmetic series, S_{98} , having $a = 1$ and $d = 1$.

$$\begin{aligned} S_{98} &= \frac{98}{2} [2(1) + 97(1)] \\ &= \frac{98 \times 99}{2} \\ &= 4851 \end{aligned}$$

From S_{98} we subtract the missing terms to obtain the sum of the given series.

$$\begin{aligned} \text{Sum} &= S_{98} - 3 - 6 - 9 - 12 - 15 - 18 - 21 - 24 - 27 - 30 - 33 \\ &\quad - 36 - 39 - 42 - 45 - 48 - 51 - 54 - 57 - 60 - 63 - 66 \\ &\quad - 69 - 72 - 75 - 78 - 81 - 84 - 87 - 90 - 93 - 96 - 99 \\ &= 4851 - 1683 \\ &= 3168 \end{aligned}$$

21. (a) $a = \frac{1}{2}$ and $r = \frac{1}{2}$

$$\begin{aligned} S_n &= \frac{\frac{1}{2} \left[\left(\frac{1}{2} \right)^n - 1 \right]}{\frac{1}{2} - 1} \\ &= \frac{\frac{1}{2} \left[\left(\frac{1}{2} \right)^n - 1 \right]}{-\frac{1}{2}} \\ &= -1 \left[\left(\frac{1}{2} \right)^n - 1 \right] \end{aligned}$$

As n gets larger and larger, $\left(\frac{1}{2} \right)^n$ gets closer and closer to 0.

Therefore, $S_{\infty} = -1(0 - 1) = 1$. The sum of the series is 1.

(b) No, if the value of r is greater than one, say $r = 2$, then the series keeps getting larger and larger as n gets larger and larger. Therefore, it is not possible to find the sum.

(c) It is possible to determine the sum of an infinite geometric series if the series converges (r is between -1 and 1).

Practice 2.4, page 128

(a) Modify the spreadsheet as shown. Use it and the SUM feature to calculate the total interest earned over the first 8 years. The table and graph are shown below.

	A	B	C	D
1	Period	Previous	Interest	Year End
2		Balance	At 4.0%/a	Balance
3	1	2000	=B3 * 0.04	=B3 + C3
4	=A3 + 1	=D3	=B4 * 0.04	=B4 + C4
			Total Interest after 8 years = \$737.14	Balance after 8 years = \$2737.14

Period	Previous Balance	Interest At 4.0%/a	Year End Balance
1	\$2,000.00	\$80.00	\$2,080.00
2	\$2,080.00	\$83.20	\$2,163.20
3	\$2,163.20	\$86.53	\$2,249.73
4	\$2,249.73	\$89.99	\$2,339.72
5	\$2,339.72	\$93.59	\$2,433.31
6	\$2,433.31	\$97.33	\$2,530.64
7	\$2,530.64	\$101.23	\$2,631.86
8	\$2,631.86	\$105.27	\$2,737.14
		\$737.14	

(b) Modify the spreadsheet as shown. Use it and the SUM feature to calculate the total interest earned over the first 8 years. The table and graph are shown below.

	A	B	C	D
1	Period	Previous	Interest	Year End
2		Balance	At 8.0%/a	Balance
3	1	8000	=B3 * 0.02	=B3 + C3
4	=A3 + 1	=D3	=B4 * 0.02	=B4 + C4
			Total Interest after 8 years = \$7076.32	Balance after 8 years = \$15 076.32